8-31

EC. Tuesday

From Friday: Surface area integrals

$$\text{Area} = \iint_S d\sigma$$

3 options:

① Parameterize:

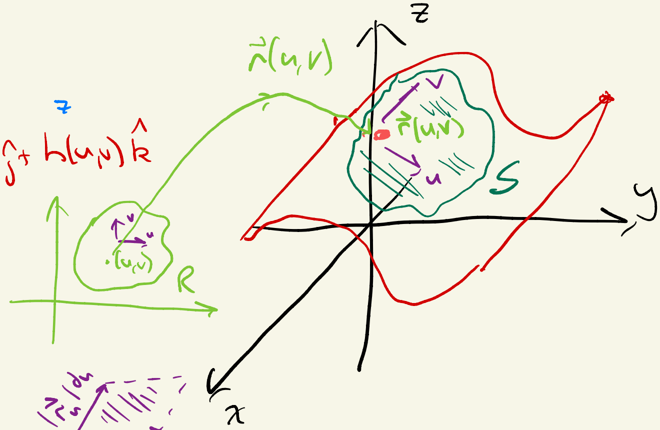
$$\vec{r}(u,v) = \overset{x}{f(u,v)}\,\hat{i} + \overset{y}{g(u,v)}\,\hat{j} + \overset{z}{h(u,v)}\,\hat{k}$$

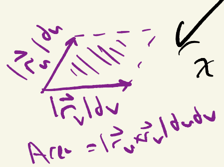
$(u,v) \in R \subseteq \mathbb{R}^2$:

$S = \vec{r}(R)$

image of R!

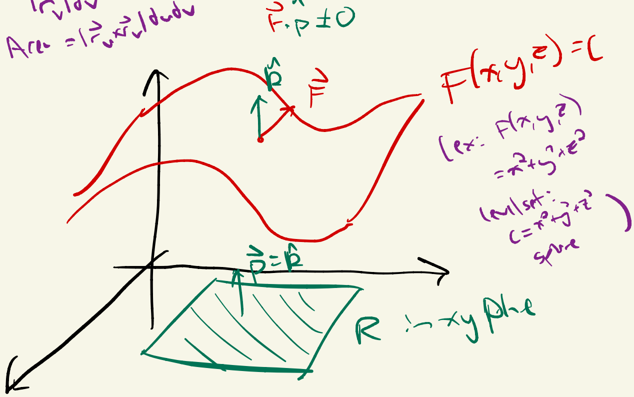
$$d\sigma = \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\text{"multidimensional " } \vec{r}\,'(t)}\, du\, dv$$





② Implicit Surface

$F(x,y,z)$ some scalar valued function. Then level sets are level surfaces $F(x,y,z) = c$



pick a plane whose normal $\hat{p}$ has $\nabla \vec{F}\cdot\hat{p} \neq 0$ over the entire region of integration

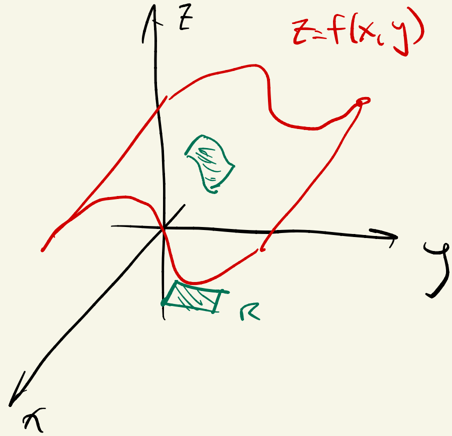
$$d\sigma = \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\, dA$$

$$\text{Area} = \iint_R \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\, dA$$

③ Explicit Graph

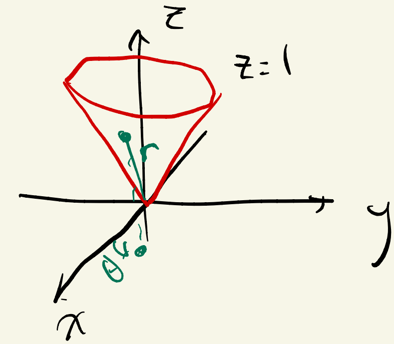
$z = f(x,y)$.

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\; dA$$



$$\text{Area} = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\; dx\,dy$$

Ex: Surface Area of cone $\quad z = r$



$$\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\hat{k}$$

$0 \le r \le 1 \leftarrow 0 \le z \le 1$

$0 \le \theta \le 2\pi$

Need: $d\sigma$

$$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + \hat{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\hat{k}$$

$$\Rightarrow \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$\hat{i}\det\begin{pmatrix} \sin\theta & 1 \\ r\cos\theta & 0 \end{pmatrix} - \hat{j}\det\begin{pmatrix} \cos\theta & 1 \\ -r\sin\theta & 0 \end{pmatrix} + \hat{k}\det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= -r\cos\theta\,\hat{i} - r\sin\theta\,\hat{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{= r}\hat{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$

$$= \sqrt{r^2 + r^2}$$

$$= \sqrt{2r^2}$$

$r \ge 0 \rightarrow$

$$= \sqrt{2}\cdot r$$

$$\Rightarrow d\sigma = \sqrt{2}\, r\, dr\, d\theta$$

Note coincidence that this looks like polar!

$$\Rightarrow \quad \text{Area} = \int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta| \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\, r \, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{2}\, d\theta$$

$$= \pi\sqrt{2}$$

↑ Think: the "flattened" the cone

area $= \pi \cdot 1^2 = \pi$

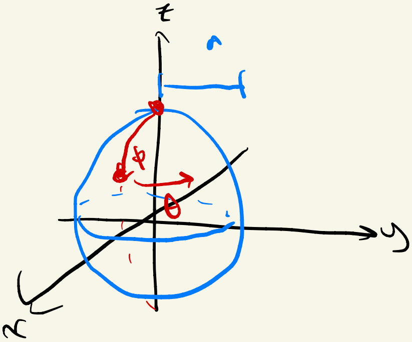
$\pi\sqrt{2} > \pi$,

as expected.

Ex: Surface area of sphere of radius a

$$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$$

$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$



First:

Find $d\sigma$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} - \left(-a^2\sin^2\phi\sin\theta\right)\hat{j}$$
$$+ \left(a^2\cos^2\theta\left(\cos\phi\sin\phi\right) + a^2\sin^2\theta\left(\sin\phi\cos\phi\right)\right)\hat{k}$$

$$= a^2\sin^2\phi\cos\theta\,\hat{i} - a^2\sin^2\phi\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k}$$

$$\Rightarrow \left|\vec{r}_\phi \times \vec{r}_\theta\right| = a^2\sqrt{\sin^2\phi}$$
$$= a^2\sin\phi \longleftarrow \quad 0 \le \phi \le \pi \Rightarrow 0 \le \sin\phi$$

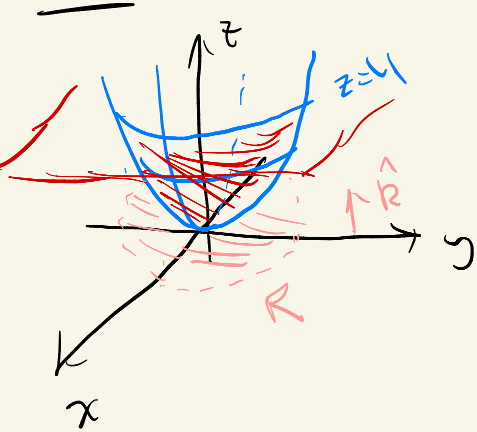
$$d\sigma = a^2\sin\phi\, d\phi\, d\theta$$
↑ is spherical coordinates!

$$\Rightarrow \text{Area} = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta$$
$$= 4\pi a^2$$

Ex:

Find the area of the surface cut from the paraboloid $z = x^2 + y^2$ by the plane $z = 4$

Soln:



want: $F(x,y,z) = c$

let $F(x,y,z) = x^2 + y^2 - z$

$\Rightarrow$ this surface is level curve for $0 = x^2 + y^2 - z$

Since $z = 4$ is cutting our paraboloid, suggests we should choose xy-plane for integration

Require: $\nabla F \cdot \hat{p} \neq 0$

$$\nabla F = \nabla(x^2 + y^2 - z)$$
$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

why not $\hat{p} = \hat{i}$?

well, $\nabla F \cdot \hat{i} = 2x$ $\leftarrow$ is 0 in our region so this won't work.

$\Rightarrow$ Chose $\hat{k} = \hat{p}$

Normal to xy plane, so region of integration R in xy.

$\Rightarrow$ using $z = 4$, R = disc $x^2 + y^2 \leq 4$

$$\nabla F = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$|\nabla F| = \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla F \cdot \hat{p}| = |\nabla F \cdot \hat{k}| = |-1| = 1$$

normal to all of R

$$\Rightarrow \text{Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\, dA$$

$$= \iint_R \frac{\sqrt{4x^2 + 4y^2 + 1}}{1}\, dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\; r\, dr\, d\theta$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

Ex: Surface area of the plane

$$\pi x + 2y + z = 2\pi$$

that lies in the first octant

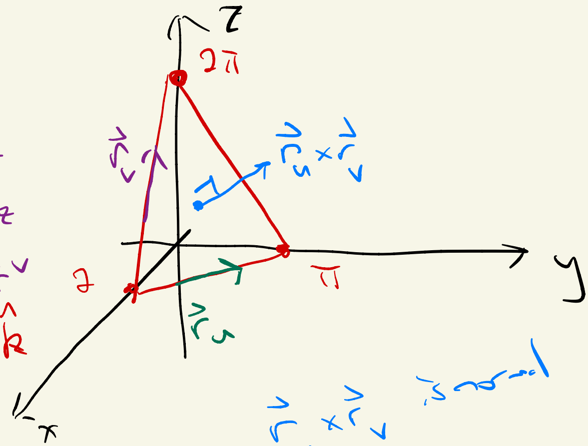
Soln:

Explicit form will be easiest: $z = 2\pi - \pi x - 2y$

But for practice, let's parametrize.

Planes can be easily parametrized. Here's one: (many correct ways of doing this)

$$u = y$$
$$v = z$$



$$\Rightarrow \quad \pi x = 2\pi - 2y - z$$
$$x = 2 - \frac{2}{\pi}y - \frac{1}{\pi}z$$
$$= 2 - \frac{2}{\pi}u - \frac{1}{\pi}v$$

$$\Rightarrow \vec{r}(u,v) = \left(2 - \frac{2}{\pi}u - \frac{1}{\pi}u\right)\hat{i} + u\hat{j} + v\hat{k}$$

$$\vec{r}_u = -\frac{2}{\pi}\hat{i} + \hat{j} + 0\hat{k}$$

$$\vec{r}_v = -\frac{1}{\pi}\hat{i} + 0\hat{j} + \hat{k}$$

$\vec{r}_u \times \vec{r}_v$ is normal to plane!

$\Rightarrow$ should be parallel to $\vec{n} = \langle \pi, 2, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{2}{\pi} & 1 & 0 \\ -\frac{1}{\pi} & 0 & 1 \end{vmatrix}$$

$$= \hat{i} - \left(-\frac{2}{\pi}\right)\hat{j} + \left(\frac{1}{\pi}\right)\hat{k}$$

$$= \hat{i} + \frac{2}{\pi}\hat{j} + \frac{1}{\pi}\hat{k}$$

notice: $\pi \cdot (\vec{r}_u \times \vec{r}_v) = \vec{n}$

$$\Rightarrow \quad |\vec{r}_u \times \vec{r}_v| = \sqrt{1 + \frac{4}{\pi^2} + \frac{1}{\pi^2}} = \sqrt{1 + \frac{5}{\pi^2}}$$

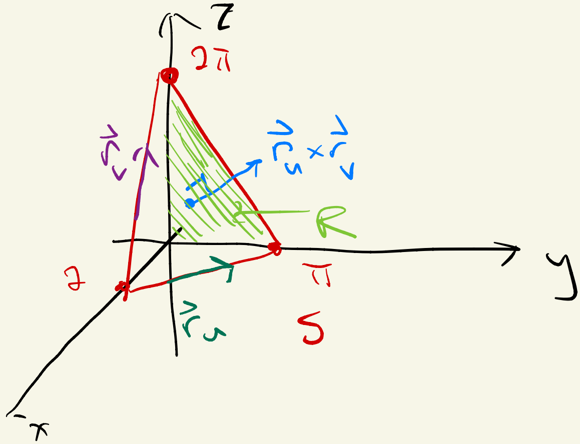
Integrating in u,v plane.

outside $u = y$

$v = z$

uv plane = yz plane

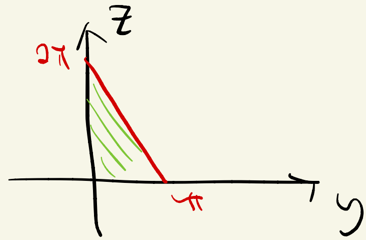
integrating in yz



$$\text{Area} = \iint_S d\sigma$$

$$= \iint_R |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$= \int_{y=0}^{y=\pi} \int_{z=0}^{z=2\pi-2y} \sqrt{1+\frac{5}{\pi^2}} \, dz\,dy$$



$$= \sqrt{1+\frac{5}{\pi^2}} \underbrace{\int_0^{\pi} \int_0^{2\pi-2y} dz\,dy}_{\text{Area of triangle}}$$

Linear thing = (something dotlike) · Area(R)

$$= \sqrt{1+\frac{5}{\pi^2}} \left(\frac{1}{2} \cdot \pi \cdot 2\pi \right)$$

$$= \pi \sqrt{\pi^2+5}$$

$\blacksquare$

$\underline{Or}$: Surface area of graph (explicit form)

$f(y,z) = x$

$$\Rightarrow x = 2 - \frac{2}{\pi}y - \frac{1}{\pi}z$$

$$\text{Area} = \iint_R \sqrt{f_y^2 + f_z^2 + 1}\, dz\, dy$$

$$= \iint_R \sqrt{\frac{4}{\pi^2} + \frac{1}{\pi^2} + 1}\, dz\, dy$$

$$= \iint_R \sqrt{\frac{5}{\pi^2} + 1}\, dz\, dy$$

$$= \pi\sqrt{\pi^2 + 5}$$

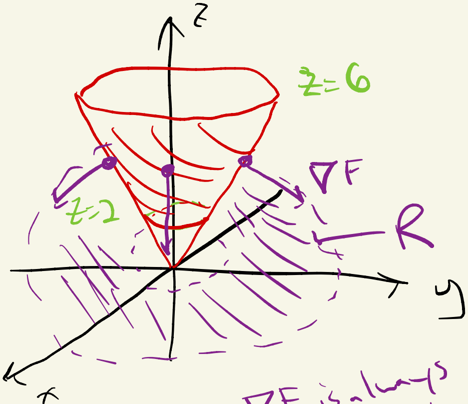
Ex: Cone frustrum (16.5 prob 19)

Surface area of portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z=2$ and $z=6$

Soln:

Could parameterize (esp. using earlier method!) or use explicit graph form, but for kicks, let's do implicit.

Want: $F(x,y,z)$ s.t. cone is level set



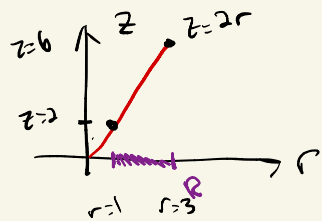
Observe: square the eqn of cone:

$z^2 = 4(x^2+y^2)$

$\Rightarrow 0 = \underbrace{4(x^2+y^2) - z^2}_{F(x,y,z)}$

∇F is always orthogonal to level sets of F

$\Rightarrow$ cone is level set for $F(x,y,z)=0$ will choose integration region so $r \geq 0$, $z \geq 0$



$\nabla F = 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$

To choose $\hat{p}$, want $\nabla F \cdot \hat{p} \neq 0$ for region $\nabla F \cdot \hat{k} = -2z \neq 0$

$\Rightarrow$ choose $\hat{p} = \hat{k}$, since $2 \leq z \leq 6$ and

$\Rightarrow$ Integrate over R in xy plane (normal to $\hat{k}$)

$\text{Area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \hat{p}|} \, dA$

$|\nabla F| = \sqrt{8^2x^2 + 8^2y^2 + 2^2z^2}$

$x^2+y^2=r^2$, cone $z=2r$

$= \sqrt{8^2r^2 + 4^2r^2}$

$= 4\sqrt{2^2r^2 + r^2} = 4r\sqrt{5}$

$|\nabla F \cdot \hat{k}| = |-2z| = |-2 \cdot 2r| = 4r$

$= \int_0^{2\pi}\int_1^3 \frac{4r\sqrt{5}}{4r} \, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_1^3 \sqrt{5} \, r\,dr\,d\theta = \boxed{8\pi\sqrt{5}}$